LESSON 20 - STUDY GUIDE

Abstract. In this lesson we will look at the other side of the pointwise convergence of Fourier series, when the partial sums either behave strangely or do not even converge at all. We will start with the well known oscillation phenomenon that occurs in the neighborhood of jump discontinuities, called the Wilbraham-Gibbs phenomenon or just Gibbs phenomenon. And in the second half of the lesson we will concentrate on the pointwise divergence of Fourier series for continuous functions. We will finish with a review of some of the most important results on almost everywhere convergence and divergence of Fourier series of functions in $L^p(\mathbb{T})$, including the celebrated theorem by Lennart Carleson on the almost everywhere pointwise convergence of Fourier series for functions in $L^2(\mathbb{T})$.

1. Pointwise convergence of Fourier series: when things do not go so well.

Study material: For the first part, on the Gibbs phenomenon, I do my own presentation but Grafakos [4] also has a section **3.5.4 - Gibbs Phenomenon** covering the same topic, although with what I believe is a slightly more complicated function and computations. For the second part of this lesson, on pointwise divergence of Fourier series of continuous functions, I follow Katznelson, in the sections **2 - Convergence and Divergence at a Point** and **3 - Sets of Divergence**, from chapter **II - The Convergence of Fourier Series**, corresponding to pgs. 51–61 in the second edition [5] and pgs. 72–82 in the third edition [6]. I also recall some functional analysis facts related to the Baire category theorem and the uniform boundedness, or Banach-Steinhaus theorem, from Folland [3] or Rudin [7].

The results from the last lesson might have given the impression that Fourier series always converge extremely well, at least for continuous functions. We had seen before, from Bernstein's theorem in Lesson 18, that the Fourier series of Hölder-α continuous functions, for $\alpha > 1/2$, converge absolutely. And in the last lesson, we concluded, from Dini's test, that for any $\alpha > 0$ the Fourier series might not converge absolutely but they still converge uniformly for all $t \in \mathbb{T}$. These are, of course, extremely strong results. Compared to Taylor series, for example, which require analyticity of the function for convergence and equality to hold in a ball of convergence, Fourier series, on the other hand, require very little, and uniform convergence on the whole domain holds for any functions with as small Hölder regularity as desired, many of which are even nowhere differentiable.

We will see in this lesson that, in spite of these results, there is still plenty of room for things to go wrong, even with continuous functions, let alone general $L^1(\mathbb{T})$ functions. We will see that there exist large sets of continuous functions for which their Fourier series diverge on uncountable dense subsets of points in $\mathbb{T}$. In fact, for many decades, it was not even known whether there could exist continuous functions whose Fourier series diverged at all points. There cannot, but this was only settled with Carleson's proof of the pointwise almost everywhere convergence of Fourier series for functions in $L^2(\mathbb{T})$, which of course contain $C(\mathbb{T})$.

For functions that are not continuous, even when pointwise converge holds at all points - for example with $f \in BV(\mathbb{T})$ - then necessarily the convergence is conditional and cannot be uniform, because uniform convergence of continuous partial sums would necessarily imply a continuous limit. Not surprisingly, things do go awry. At jump discontinuities, for example, where we know that the Fourier series converges to the average of lateral limits at the jump, the partial sums nevertheless oscillate strongly in order

Date: May 7, 2020.

to represent the jump. And because there is no uniform convergence, this oscillation never disappears, exhibiting a ringing, or overshoot, phenomenon first discovered by Henry Wilbraham, in 1848, and then independently in 1899 by J. Willard Gibbs, which typically carries the latter's name. We will start this lesson with it.

1.1. **The Wilbraham-Gibbs phenomenon.** The best model function to study the Wilbraham-Gibbs phenomenon, usually known simply as the Gibbs phenomenon, is to investigate the convergence in a neighborhood of the origin of the Fourier series associated to a Heaviside-type jump,

$$f(t) = \begin{cases} -1 & \text{for} \quad -\pi < t < 0, \\ +1 & \text{for} \quad 0 \le t \le \pi. \end{cases} \tag{1.1}$$

This function is obviously of bounded variation, its total variation over $\mathbb{T}$ being 4, with two jump discontinuities at $t = 0$ and $t = \pi$, where the Fourier series necessarily converges to zero. Everywhere else the Fourier series converges to the actual values of the function, 1 or -1, uniformly on compact subsets of the intervals $]-\pi, 0[$ and $]0, \pi[$. Thus

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{int} = \begin{cases} -1 & \text{for} \quad -\pi < t < 0, \\ +1 & \text{for} \quad 0 < t < \pi, \\ 0 & \text{for} \quad t = 0, \pi. \end{cases}$$

Due to the Riemann localization principle, Proposition 1.8 in Lesson 19, the Fourier series of any piecewise smooth function will exhibit the same behavior (with obvious rescaling and translation) at any of its jump discontinuities as this example at $t = 0$.

So, writing the partial sums of the Fourier series of f in (1.1) as the convolution with the Dirichlet kernel, we have

$$\begin{aligned} S_N[f](t) = \sum_{n=-N}^{N} \hat{f}(n) e^{int} &= D_N * f(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(t-s) f(s) ds \\ &= \frac{1}{2\pi} \int_0^{\pi} D_N(t-s) ds - \frac{1}{2\pi} \int_{-\pi}^{0} D_N(t-s) ds \\ &= \frac{1}{2\pi} \int_{t-\pi}^{t} D_N(s) ds - \frac{1}{2\pi} \int_{t}^{t+\pi} D_N(s) ds \\ &= \frac{1}{2\pi} \int_{-t}^{-t+\pi} D_N(s) ds - \frac{1}{2\pi} \int_{t}^{t+\pi} D_N(s) ds \\ &= \frac{1}{2\pi} \int_{-t}^{t} D_N(s) ds + \frac{1}{2\pi} \int_{t}^{-t+\pi} D_N(s) ds - \frac{1}{2\pi} \int_{t}^{t+\pi} D_N(s) ds \\ &= \frac{1}{2\pi} \int_{-t}^{t} D_N(s) ds - \frac{1}{2\pi} \int_{-t+\pi}^{t+\pi} D_N(s) ds, \end{aligned}$$

where we used the fact that the Dirichlet kernel is an even function, to perform these computations. We then observe that the partial sums of the Fourier series of f consist of the difference of two integrals of the Dirichlet kernel: one in a neighborhood of the origin $s = 0$, from $s = -t$ to $s = t$, and the other in a neighborhood of $s = \pi$, from $s = \pi - t$ to $s = \pi + t$. The Dirichlet kernel, written as

$$D_N(s) = \frac{\sin(N + \frac{1}{2})s}{\sin \frac{s}{2}},$$

is only singular at the origin. So the second integral above, around $s=\pi$, can be easily shown to converge to zero with the now usual oscillatory integral/Riemann-Lebesgue argument. In fact

$$\begin{aligned}\frac{1}{2\pi}\int_{\pi-t}^{\pi+t} D_N(s)ds &= \frac{1}{2\pi}\int_{\pi-t}^{\pi+t} \frac{\sin(N+\frac{1}{2})s}{\sin\frac{s}{2}}ds \\ &= \frac{1}{2\pi}\int_{\mathbb{T}} \frac{\sin(N+\frac{1}{2})s}{\sin\frac{s}{2}}\chi_{[\pi-t,\pi+t]}(s)ds \\ &= \frac{1}{2\pi}\int_{\mathbb{T}} \frac{\sin(Ns)}{\tan\frac{s}{2}}\chi_{[\pi-t,\pi+t]}(s)ds + \frac{1}{2\pi}\int_{\mathbb{T}} \cos(Ns)\,\chi_{[\pi-t,\pi+t]}(s)ds,\end{aligned}$$

where $\chi_{[\pi-t,\pi+t]}$ is the characteristic function of the interval $[\pi-t,\pi+t]$. So if we consider a compact interval of values of t, say $|t|\le\pi/2$, then both $\frac{1}{\tan\frac{s}{2}}\chi_{[\pi-t,\pi+t]}(s)$ and $\chi_{[\pi-t,\pi+t]}(s)$ form a compact family of functions in $L^1(\mathbb{T})$ and, as discussed in the previous lesson for the pointwise convergence of the partial sums of Fourier series with Dini's test, the Riemann-Lebesgue lemma holds uniformly and therefore

$$\frac{1}{2\pi}\int_{\pi-t}^{\pi+t} D_N(s)ds \to 0, \quad \text{as} \quad N\to\infty,$$

uniformly for $|t|\le\pi/2$. So,

$$S_N[f](t) = \sum_{n=-N}^{N}\hat{f}(n)e^{int} = \frac{1}{2\pi}\int_{-t}^{t} D_N(s)ds + o(1) = \frac{1}{2\pi}\int_{-t}^{t}\frac{\sin(N+\frac{1}{2})s}{\sin\frac{s}{2}}ds.$$

Again, also as we did in the last lesson, we replace the $\sin\frac{s}{2}$ term in the denominator by $s/2$ by observing that

$$\left(\frac{1}{\sin\frac{s}{2}} - \frac{1}{\frac{s}{2}}\right) \in L^\infty(\mathbb{T}),$$

and thus

$$\begin{aligned}S_N[f](t) &= \sum_{n=-N}^{N}\hat{f}(n)e^{int} = \frac{1}{2\pi}\int_{-t}^{t}\frac{\sin(N+\frac{1}{2})s}{\frac{s}{2}}ds \\ &= \frac{2}{\pi}\int_0^t \frac{\sin(N+\frac{1}{2})s}{s}ds \\ &= \frac{2}{\pi}\int_0^{(N+\frac{1}{2})t}\frac{\sin s}{s}ds.\end{aligned}$$

Observe now that the maximum value of this integral occurs at the first root of $\sin s$, i.e. when $(N+\frac{1}{2})t=\pi$, with value

$$\frac{2}{\pi}\int_0^\pi \frac{\sin s}{s}ds = 1,089490\ldots$$

and we conclude that, at the jump discontinuity, there is an overshoot of the function of about 9% at the first oscillation, on symmetric sides of the jump. And the only thing that happens, as $N\to\infty$ is that this first oscillation moves closer to the discontinuity, at a rate of $t=\pi/(N+\frac{1}{2})$ but does not decrease its height, let alone disappear. The following figures, for $N=5, 25$ and 125 clearly illustrate how this convergence evolves as N increases, in a classical example of pointwise convergence that is not uniform.

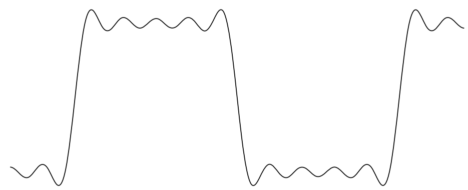

Figure 1. N=5

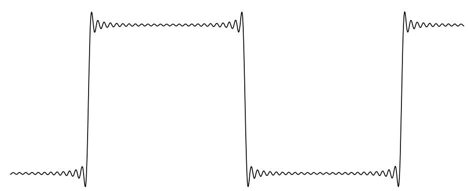

Figure 2. N=25

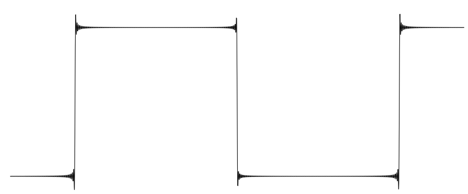

Figure 3. N=125

1.2. **Pointwise divergence of Fourier series of continuous functions.** To see that there exist plenty of continuous functions with divergent Fourier series we will use a powerful combination of explicit estimates with a deep theorem of Functional Analysis: the uniform boundedness, or Banach-Steinhaus, theorem.

Let us start by considering the sequence of linear functionals that take each function $f \in C(\mathbb{T})$ to the value of the Nth partial sum of its Fourier series at the origin:

$$f \in C(\mathbb{T}) \mapsto S_N[f](0) \in \mathbb{C}.$$

As the partial sums are given by the convolution with the Dirichlet kernel, a simple convolution estimate yields boundedness of each of these functionals

$$|S_N[f](0)| \leq \|S_N[f]\|_{L^\infty(\mathbb{T})} = \|D_N * f\|_{L^\infty(\mathbb{T})} \leq \|D_N\|_{L^1(\mathbb{T})} \|f\|_{L^\infty(\mathbb{T})},$$

where, for continuous functions, $\|f\|_{L^\infty} = \|f\|_{C(\mathbb{T})} = \sup_{t\in\mathbb{T}} |f(t)|$. So these linear functionals have operator norms bounded by the Lebesgue constants

$$\|S_N[\cdot](0)\|_{C(\mathbb{T})\to\mathbb{C}} \leq \|D_N\|_{L^1(\mathbb{T})}.$$

We will now see that the norms are, not only bounded, but exactly equal to the Lebesgue constants $\|D_N\|_{L^1(\mathbb{T})}$. In fact

$$|S_N[f](0)| = |D_N * f(0)| = \left|\frac{1}{2\pi}\int_{\mathbb{T}} D_N(-t)f(t)dt\right| = \left|\frac{1}{2\pi}\int_{\mathbb{T}} D_N(t)f(t)dt\right|,$$

because D_N is even. But, just as we did back in the lessons about L^p spaces and duality, we can now take $f \in L^\infty(\mathbb{T})$ of the form $f(t) = \operatorname{sgn} D_N(t)$, i.e.

$$f(t) = \begin{cases} 1 & \text{if} \quad D_N(t) > 0 \\ 0 & \text{if} \quad D_N(t) = 0 \\ -1 & \text{if} \quad D_N(t) < 0 \end{cases}$$

This, of course, has unit norm $\|f\|_{L^\infty(\mathbb{T})} = 1$ and shows that the same linear functional on the slightly larger domain $L^\infty(\mathbb{T})$ does have operator norm equal to the Lebesgue constants

$$\|S_N[\cdot](0)\|_{L^\infty(\mathbb{T})\to\mathbb{C}} = \sup_{\|f\|_{L^\infty(\mathbb{T})}=1} |S_N[f](0)| = \|D_N\|_{L^1(\mathbb{T})}.$$

It is now trivial to modify this function f at the finite points of jump discontinuities to make it continuous (for example by transforming the sharp jumps into steep slopes) in order to make the integral above arbitrarily close to $\|D_N\|_{L^1(\mathbb{T})}$, by just using continuous functions f with supremum equal to one. We thus obtain also

$$\|S_N[\cdot](0)\|_{C(\mathbb{T})\to\mathbb{C}} = \sup_{\|f\|_{C(\mathbb{T})}=\|f\|_{L^\infty(\mathbb{T})}=1} |S_N[f](0)| = \|D_N\|_{L^1(\mathbb{T})}. \tag{1.2}$$

At this point, we once again run into the importance of estimating the L^1 norms of the Dirichlet kernel, which was left as an exercise to show that

$$\|D_N\|_{L^1(\mathbb{T})} = \frac{4}{\pi^2}\log N + O(1).$$

The unboundedness of these Lebesgue constants provided confirmation, a few lessons ago, that the Dirichlet kernel is not an approximate identity. Now it shows us that, for each N, we can pick a continuous function f_N, with unit supremum, for which the corresponding partial sums of its Fourier series satisfy

$$|S_N[f_N](0)| \sim \frac{4}{\pi^2}\log N.$$

These partial sums are unbounded, but as N changes so does the function f_N. We would like to be able to pick a single continuous function, for which all of its partial sums satisfy this same estimate, therefore obtaining a function whose partial sums at $t = 0$ do not converge.

Such a function can be constructed explicitly by using the so-called *gliding hump* method. We'll just briefly sketch the idea, because we'll use a another method right afterwards, based on Baire's category theorem and the Banach-Steinhaus theorem of functional analysis, that gives us a more general result.

So, because the Lebesgue constants are unbounded, for each $l \in \mathbb{N}$ we can chose an order N_l such that

$$\|S_{N_l}[\cdot](0)\|_{C(\mathbb{T})\to\mathbb{C}} > 2^l.$$

We explained above how to obtain continuous functions f_{N_l}, with unit supremum, for which their partial sums of order N_l satisfy this inequality at the origin. But because trigonometric polynomials are dense in $C(\mathbb{T})$ we can actually use a polynomial P_l in the place of f_{N_l}, so that

$$|S_{N_l}[P_l](0)| > 2^l, \qquad \text{with} \quad \sup_{t\in\mathbb{T}} |P_l(t)| = 1.$$

Clearly, degree(P_l) $> N_l$, for otherwise the partial sums of order N_l would coincide with the polynomial itself and we would have $|S_{N_l}[P_l](0)| = |P_l(0)| \leq 1$, which would not satisfy the 2^l lower bound. The trigonometric polynomials have bounded support in the frequencies, with non-zero coefficients only between $-$degree(P_l) and $+$degree(P_l). The goal then is to construct the final function by summing translates of each of these polynomials in frequency, so that their supports in Fourier space do not overlap. We do that multiplying the polynomials by exponentials $e^{iM_l t}$, for which the support in frequency of

$$e^{iM_l t} P_l(t),$$

now sits between $M_l -$degree(P_l) and $M_l +$degree(P_l). And if we carefully pick the sequence of increasing frequencies M_l, where the successive supports of the polynomials are centered, so that they do not overlap - thus the gliding or sliding hump - then we obtain the desired function

$$f(t) = \sum_{l=1}^{\infty} \frac{1}{l^2}\, e^{iM_l t}\, P_l(t).$$

This function is obviously continuous, and because the polynomials P_l have partial sums at the origin bounded below by 2^l, the partial sums of f are also unbounded at the origin and its Fourier series diverges. The details can be found in Katznelson's book, section **2 - Convergence and Divergence at a Point** from chapter **II - The Convergence of Fourier Series**, corresponding to pgs. 51–52 in the second edition [5] and pgs. 72–74 in the third edition [6].

This was what could be called a hard analysis approach to constructing the example of a continuous function whose Fourier series diverges at the origin. Let us now use a soft analysis argument which, in this case, provides a much more general answer. It is based on the uniform boundedness, or Banach-Steinhaus theorem, which in turn is usually proved as a consequence of the Baire category theorem. We recall both, now (for details and proofs, see Folland's book [3], section **5.3 - The Baire Category Theorem and its Consequences** from chapter **5 - Elements of Functional Analysis**, or Rudin [7], section **Consequences of Baire's Theorem** from chapter **5 - Examples of Banach Space Techniques**).

Theorem 1.1. **(Baire)** *Let X be a complete metric space. If $\{O_n\}_{n\in\mathbb{N}}$ is a countable collection of open and dense subsets of X then $\cap_{n\in\mathbb{N}} O_n$ is dense in X.*

The Baire category theorem is usually stated by saying that a complete metric space cannot be the countable union of nowhere dense sets, which is just a consequence of the previous statement, by thinking of the complements of the open sets. The advantage of stating it as above is that we have a more precise description that what is left from the intersection of a countable collection of open dense sets is still a dense set, a G_δ[1]. We can even go further.

Corollary 1.2. *In a complete metric space, the intersection of a countable collection of dense G_δ sets is still a dense G_δ set.*

From the Baire theorem, the uniform boundedness theorem for families of linear bounded operators follows.

Theorem 1.3. **(Banach-Steinhaus)** *Let X be a Banach space, Y a normed vector spaces and $\mathcal{T}$ any set of bounded linear maps from X to Y. Then, either the collection $\mathcal{T}$ is uniformly bounded in their operator norms,*

$$\sup_{T\in\mathcal{T}} \|T\|_{X\to Y} < \infty$$

or there is a dense G_δ of points $x \in X$ for which

$$\sup_{T\in\mathcal{T}} \|Tx\|_Y = \infty.$$

[1] Recall that a G_δ set is a set obtained by the countable intersection of open sets.

This theorem is also more typically stated as saying that, if the family $\mathcal{T}$ is pointwise bounded, i.e. if for every $x \in X$ there exists a constant $C_x \geq 0$ such that

$$\sup_{T \in \mathcal{T}} \|Tx\|_Y \leq C_x,$$

then the first alternative holds, and the collection $\mathcal{T}$ is uniformly bounded. Again, for us it is more useful to state the theorem the way we did, because it describes more precisely what happens when a family of bounded linear operators is *not* uniformly bounded: the pointwise bounds don't just fail at one point, but more dramatically, at a whole dense G_δ subset of points of X.

The way that these theorems can be applied to our search for continuous functions whose Fourier series diverge at a point should now be obvious. From (1.2) we know that the operator norms of the partial sums at the origin are *not* uniformly bounded, for they are equal to the Lebesgue constants $\|D_N\|_{L^1(\mathbb{T})}$ which grow as $\log N$. Consequently, from the Banach-Steinhaus theorem above, there must exist a dense G_δ set of continuous functions for which

$$\sup_N |S_N[f](0)| = \infty,$$

and therefore whose partial sums at the origin diverge.

But we can expand this result even more. We now take any countable set of points $\{t_i\}_{i\in\mathbb{N}}$ in $\mathbb{T}$ and repeat the above argument for each of them, thus obtaining for each one a G_δ dense subset of the continuous functions $E_i \subset C(\mathbb{T})$ whose Fourier series diverge at t_i. And from Corollary 1.2 we conclude that $E = \cap_i E_i$ is still a G_δ dense subset of $C(\mathbb{T})$ all of whose functions have Fourier series that diverge at all points $\{t_i\}_{i\in\mathbb{N}}$.

To finish this line of argument, we observe that, for each continuous f, the set of points where its Fourier series diverges is also a G_δ subset of $\mathbb{T}$ because

$$\{t \in T : \sup_N |S_N[f](t)| = \infty\} = \cap_k \{t \in T : \sup_N |S_N[f](t)| > k\},$$

and the sets that make up the intersection are open in $\mathbb{T}$. So if we pick the countable set of points $\{t_i\}_{i\in\mathbb{N}}$ in $\mathbb{T}$ so that they are dense, for example by choosing them to be the rationals on the circle, than we finally conclude with the very striking following result.

Theorem 1.4. (duBois Reymond) *There exists a dense G_δ subset of continuous functions $E \subset C(\mathbb{T})$ such that each of its functions $f \in E$ has a Fourier series that diverges on a dense G_δ subset of points of $\mathbb{T}$.*

This result becomes even more impressive if we realize that none of these G_δ sets can be countable because on complete metric spaces without isolated points, a countable set can never be a G_δ (see Rudin [7]). We have thus proved the existence of an uncountable and dense set of continuous functions whose Fourier series each diverge at uncountable dense sets of points of $\mathbb{T}$.

So, despite the very positive results that we saw in the last lesson about the pointwise convergence, even uniform, of Fourier series of continuous functions with as little Hölder regularity as desired, there are still many continuous functions left, in fact uncountable dense sets of them, whose Fourier series diverge at uncountable dense sets of points of the circle.

Yitzhak Katznelson, himself an expert on divergence of Fourier series, has a full section **3 - Sets of Divergence** from chapter **II - The Convergence of Fourier Series** in his book [5, 6] totally dedicated to fine issues of pointwise divergence. Just for the sake of knowing the existence of some of these results, let us review the most important of them very briefly, without getting into details or proofs. The main concept is that of *set of divergence* of $L^p(\mathbb{T})$ or $C(\mathbb{T})$.

Definition 1.5. *A set $E \subset \mathbb{T}$ is called a set of divergence of one of the Banach spaces $L^p(\mathbb{T})$ or $C(\mathbb{T})$ if there exists a function in this space for which the corresponding partial sums of the Fourier series diverge at every point of E.*

A first interesting fact is that countable unions of sets of divergence still form sets of divergence.

Theorem 1.6. *Let $E_{i i\in\mathbb{N}}$ be a countable collection of sets of divergence for one of the Banach spaces $L^p(\mathbb{T})$ or $C(\mathbb{T})$. Then, $E = \cup_{i\in\mathbb{N}} E_i$ is a set of divergence.*

In 1965, Jean-Pierre Kahane and Yitzhak Katznelson proved the following.

Theorem 1.7. (Kahane-Katznelson) *Every set of Lebesgue measure zero in $\mathbb{T}$ is a set of divergence for $C(\mathbb{T})$.*

So, given any zero measure set, it is always possible to pick a particular continuous function whose Fourier series diverges at every point of that set.

Finally, the next theorem states a truly remarkable and surprising fact.

Theorem 1.8. (Kahane-Katznelson) *For the Banach spaces $L^p(\mathbb{T})$, $1 \leq p \leq \infty$, or $C(\mathbb{T})$, either the whole circle $\mathbb{T}$ is a set of divergence, or the sets of divergence are just the sets of measure zero.*

So, fixing each of the $L^p(\mathbb{T})$ spaces, or $C(\mathbb{T})$, one either finds functions with Fourier series that diverge everywhere, or the Fourier series of all functions converge pointwise almost everywhere. There is no intermediate situation, where there are functions whose Fourier series diverge on sets of positive measure, but none that diverges everywhere.

Given this amazing subtlety of situations, it is no wonder that at the beginning of the twentieth century, after some solid acquaintance had been developed with the newly created Lebesgue theory of integration and many of these results had been proved, serious doubts were raised as to the pointwise convergence of Fourier series of $L^p(\mathbb{T})$ functions, and even continuous functions, that, for many years throughout the nineteenth century were believed to hold everywhere.

In 1915, Nikolai Luzin conjectured that for functions in $L^2(\mathbb{T})$ their Fourier series would converge almost everywhere, and this would obviously include continuous functions. But this remained an important open problem for many years. In another surprising discovery, Kolmogorov, who had been Luzin's PhD student, proved in 1923 that there exist functions in $L^1(\mathbb{T})$ whose Fourier series diverge everywhere.

Theorem 1.9. (Kolmogorov) *The whole circle $\mathbb{T}$ is a set of divergence for $L^1(\mathbb{T})$.*

Luzin's conjecture was famously settled only in 1965 by Lennart Carleson, and published in a paper the following year [2], which also finally answered the analogous question for continuous functions, that had remained unanswered until that point. In fact, Carleson has declared in interviews that he had initially started working on the problem with the goal of finding a continuous function with an everywhere divergent Fourier series, that people at that point were starting to believe to be the truth. And only halfway through his work did he realize that he could actually prove the opposite fact.

Theorem 1.10. (Carleson [2]) *Let $f \in L^2(\mathbb{T})$. Then the symmetric partial sums of its Fourier series $\sum_{-N}^{N} \hat{f}(n)e^{int}$ converge pointwise almost everywhere $t \in \mathbb{T}$ to $f(t)$ as $N \to \infty$.*

Within a couple of years from Carleson's proof, Richard Hunt extended it to any $L^p(\mathbb{T})$, with $p > 1$. And in 1973, Charles Fefferman provided a different proof of the pointwise almost everywhere convergence of Fourier series of functions in $L^p(\mathbb{T})$, with $p > 1$.

The proofs of the results, down to the construction of the $L^1(\mathbb{T})$ function with an everywhere divergent Fourier series can all be found in Katznelson's book [5, 6]. Carleson's proof is famous for being extremely difficult and technical. For those interested in it, I suggest a monograph by Juan Arias de Reyna [1], that he wrote after trying to understand Carlson's proof. It is a detailed and mostly self contained analysis of Carleson's theorem and of the mathematical analysis tools associated with its proof.

References

[1] Juan Arias de Reyna, *Pointwise convergence of Fourier series,* Springer Lecture Notes in Mathematics, vol. 1785, 2002.
[2] Lennart Carleson, *On convergence and growth of partial sums of Fourier series,* Acta Mathematica, 116 (1), 135–157, 1966
[3] Gerald B. Folland, *Real Analysis, Modern Techniques and Applications,* 2nd Edition, John Wiley & Sons, 1999.
[4] Loukas Grafakos, *Classical Fourier Analysis,* 3rd Edition, Springer, Graduate Texts in Mathematics 249, 2014.
[5] Yitzhak Katznelson *An Introduction to Harmonic Analysis,* 2nd Edition, Dover Publications, 1976.
[6] Yitzhak Katznelson *An Introduction to Harmonic Analysis,* 3rd Edition, Cambridge University Press, 2004.
[7] Walter Rudin *Real and Complex Analysis,* 3rd Edition, McGraw-Hill, 1987.